Models for neutrino masses and mixing

Introduction

Introductory remarks:

- No problem to accommodate neutrino masses and mixing
- The problem is to explain characteristic features
- Mass and mixing problem in the quark sector unexplained as well
- It could be that the mass problem is decoupled from mixing problem, i.e., perhaps one can find models which explain mixing but not the masses, mass problem maybe more fundamental

ν masses and mixing can be accommodated, e.g., in
SM + 3 ν_R + L conserv. $\to$ complete analogy to quark sector
(Dirac neutrinos)

Issues one would like to understand:

1. Why are ν masses much smaller than charged lepton masses?

   Two "proposals" for a solution:

   - Seesaw mechanism
   - Radiative neutrino masses

2. Can one reproduce the special features of ν masses and mixing?

F1 $\theta_\odot \simeq 34^\circ\,{}^{+3^\circ}_{-2^\circ}$ (90% CL) (large but non-maximal)

F2 $\theta_{\rm atm}$ large (maximal ???) [$\theta_{\rm atm} \simeq 45^\circ \pm 8^\circ$ (90% CL)]

F3 $|U_{e3}|^2 \equiv s_{13}^2 \lesssim 0.02$ (90% CL)

F4 $\Delta m^2_\odot / \Delta m^2_{\rm atm} \sim 0.03$

Lepton mixing matrix:

$$\boxed{U = U_{23}U_{13}U_{12}}$$

atm/LBL
ν_μ disapp.
$\nu_\mu \leftrightarrow \nu_\tau$

$$U_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$

atm/LBL
ν_e disapp.
$\nu_e \leftrightarrow \nu_\mu$

$$U_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}$$

solar ν_e
KamLAND:
$\bar{\nu}_e$ disapp.

$$U_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Phase δ analogous to CKM phase

Framework:

- Simple extensions of the lepton sector of the SM, i.e. the gauge group is $G = SU(2)_L \times U(1)_Y$
- Extension of the fermion sector:
  right-handed neutrino singlets $\nu_R \to$ seesaw mechanism
- All possible extensions of the scalar sector
- Flavour symmetries for enforcing certain features of the PMNS matrix
- Emphasis on Majorana neutrinos

Mass term for Majorana neutrinos: $\boxed{\mathcal{L}_{\mathrm{maj}} = \frac{1}{2}\, \nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \mathrm{H.c.}}$

Charge conjugation matrix properties

C with $C^{-1}\gamma^\mu C = -\left(\gamma^\mu\right)^T$, $C^T = -C$, $C^\dagger = C^{-1}$

$\mathcal{L}_{\mathrm{maj}}$ Lorentz-invariant, $\boxed{\mathcal{M}_\nu^T = \mathcal{M}_\nu}$ ($\mathcal{M}_\nu$ complex symmetric)

Diagonalization of $\mathcal{M}_\nu$: Schur 1945

$$V^T \mathcal{M}_\nu V = \hat{m} \equiv \mathrm{diag}\,(m_1, m_2, m_3)$$
$$V = e^{i\hat{\varphi}}\, U\, \mathrm{diag}\,(e^{i\rho}, e^{i\sigma}, 1)$$

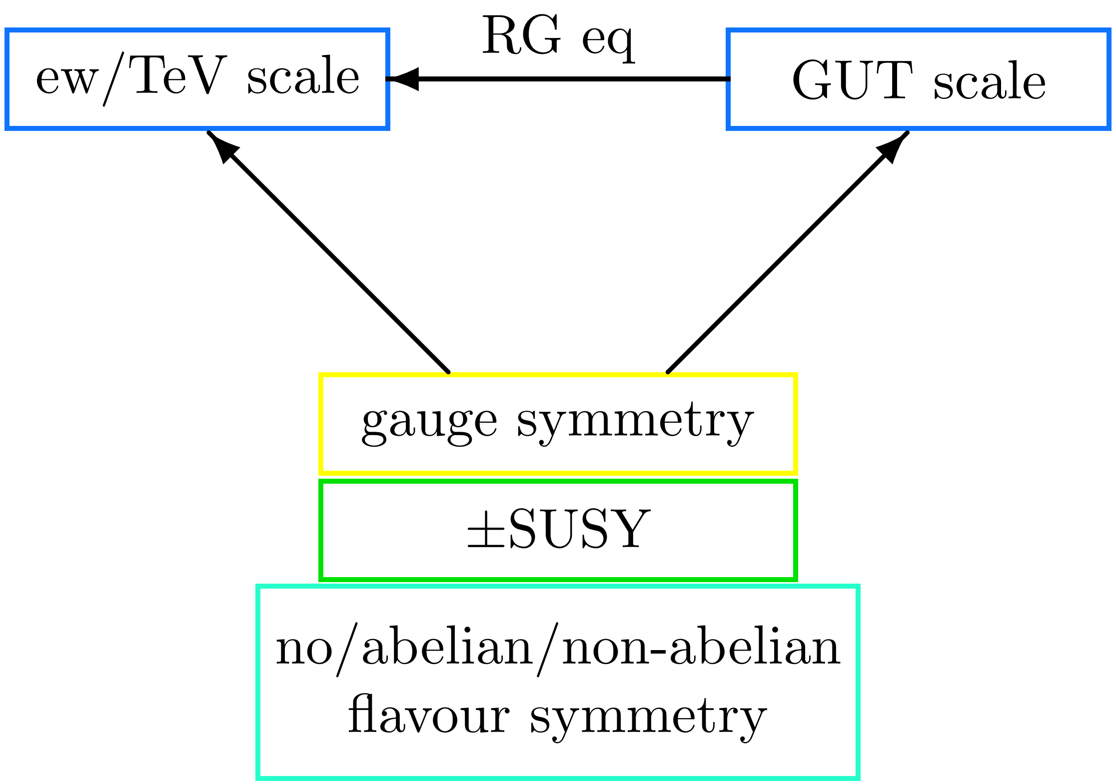
Diagonal phase matrix $e^{i\hat{\varphi}}$

Basis where charged lepton mass matrix M_ℓ is diagonal:
$e^{i\hat{\varphi}} = \mathrm{diag}\left(e^{i\varphi_e}, e^{i\varphi_\mu}, e^{i\varphi_\tau}\right)$ unphysical because can be absorbed into the left-handed charged lepton fields

$$-\mathcal{L}_{cc}: \quad \frac{g}{\sqrt{2}} W_\mu^- \bar{\ell}_L \gamma^\mu \nu_L = \frac{g}{\sqrt{2}} W_\mu^- \bar{\ell}_L \gamma^\mu V \hat{\nu}_L$$

Mass eigenfields $\hat{\nu}_L$
Define new fields $\ell'_L = e^{-i\hat{\varphi}} \ell_L \Rightarrow e^{i\hat{\varphi}}$ removed from $\mathcal{L}_{cc}$
Note: absorbing phases different from $e^{i\hat{\varphi}}$ into $\ell_L \rightarrow$ phase convention in U different from standard one
If M_ℓ NOT diagonal, $e^{i\hat{\varphi}}$ has physical effects

Myriads of textures/models!



* Is the explanation of F1-4 independent of general fermion mass problem?

* How close to 1 is $\sin^2 2\theta_{\rm atm}$? (more accurate: θ_{23} close to $45°$?) $\rightarrow$ non-abelian flavour symmetry?

Extensions of the SM

(a) Right-handed neutrino singlets

Multiplets:

	$SU(2)$	$\times\ U(1)$	
D_L	$\underline{\frac{1}{2}}$	$Y = -1$	left-handed doublets
ℓ_R	$\underline{0}$	$Y = -2$	right-handed charged lepton singlets
ν_R	$\underline{0}$	$Y = 0$	right-handed ν singlets
ϕ	$\underline{\frac{1}{2}}$	$Y = 1$	Higgs doublet
$\tilde{\phi}$	$\underline{\frac{1}{2}}$	$Y = -1$	Higgs doublet

$$\tilde{\phi} \equiv i\tau_2 \phi^* \Rightarrow \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \leftrightarrow \tilde{\phi} = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$$

Motivation of ν_R by $SO(10)$ GUT:
chiral fields per family =
$2 \times 2 \times 3$ (quarks: up, down; L, R; coulor) $+ 2 \times 2$ (leptons: ℓ, ν; L, R) $= 16$
$\rightarrow$ fit exactly into 16-plet of $so(10)$ with correct quantum numbers
$so(10) =$ Lie algebra of $SO(10)$ and its covering group $Spin(10)$

Lagrangian: SM + $3\,\nu_R$ + L violation

Remarks: Could also choose 2 or more than 3 ν_R, allow for an arbitrary number of Higgs doublets

$$\begin{aligned} \mathcal{L} &= \cdots - \textstyle\sum_j \left[\bar{\ell}_R \phi_j^\dagger \Gamma_j + \bar{\nu}_R \tilde{\phi}^\dagger \Delta_j \right] D_L + \text{H.c.} \\ &\quad + \left(\tfrac{1}{2}\, \nu_R^T C^{-1} M_R^* \nu_R + \text{H.c.} \right) \qquad \boxed{M_R = M_R^T} \end{aligned}$$

$$\langle \phi_j^0 \rangle_0 = v_j \;\Rightarrow\; M_\ell = \sum_j v_j^* \Gamma_j \,, \quad M_D = \sum_j v_j \Delta_j$$

Total Maj. mass term for left-handed ν fields:

$$\mathcal{L}_{\nu\,\text{mass}} = \tfrac{1}{2}\, \omega_L^T C^{-1} \mathcal{M}_{D+M}\, \omega_L + \text{H.c.}$$

$$\mathcal{M}_{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \quad \text{with} \quad \omega_L = \begin{pmatrix} \nu_L \\ C(\bar{\nu}_R)^T \end{pmatrix}$$

Remark: L conservation $\Rightarrow M_R = 0$, Dirac ν masses in complete analogy to other fermion masses, no explanation for $m_\nu \ll m_\ell$

Derivation of $\mathcal{L}_{\nu\,\mathrm{mass}}$:

Charge-conjugate fields: $(\psi_R)^c \equiv C(\bar{\psi}_R)^T = C{\gamma^0}^T \psi_R^*$ left-handed

Note: $(\psi_R)^{cc} = \psi_R$ or $\psi_R = C{\gamma^0}^T[(\psi_R)^c]^*$ or $\psi_R^* = -C^{-1}\gamma^0(\psi_R)^c$

${\gamma^0}^T C^{-1}\gamma^0 = -C^{-1}$

Dirac term:

$$-\bar{\nu}_R M_D \nu_L = -\left(C{\gamma^0}^T[(\nu_R)^c]^*\right)^\dagger \gamma^0 M_D \nu_L = [(\nu_R)^c]^T C^{-1} M_D \nu_L$$

$$= \frac{1}{2}\left\{[(\nu_R)^c]^T C^{-1} M_D \nu_L + \nu_L^T C^{-1} M_D^T [(\nu_R)^c]\right\}$$

Majorana Term:

$$\left(\frac{1}{2}\,\nu_R^T C^{-1} M_R^* \nu_R\right)^\dagger = \frac{1}{2}\,\nu_R^\dagger C M_R \nu_R^* =$$

$$\frac{1}{2}\,[-C^{-1}\gamma^0(\nu_R)^c]^T C M_R [-C^{-1}\gamma^0(\nu_R)^c] =$$

$$\frac{1}{2}\,[(\nu_R)^c]^T\left(-{\gamma^0}^T C^{-1}\gamma^0\right) M_R(\nu_R)^c = \frac{1}{2}\,[(\nu_R)^c]^T C^{-1} M_R(\nu_R)^c$$

Seesaw mechanism: Minkowski 1977 [1]

Assumption: $m_D \ll m_R$ ($m_{D,R}$ scales of $M_{D,R}$)

More precise:

Largest eigenvalue of $\sqrt{M_D^\dagger M_D} \ll$ smallest eigenvalue of $\sqrt{M_R^\dagger M_R}$

Wanted: unitary 6×6 matrix W which disentangles small from large scale $\rightarrow$ ansatz [2]:

$$W = \begin{pmatrix} \sqrt{\mathbb{1} - BB^\dagger} & B \\ -B^\dagger & \sqrt{\mathbb{1} - B^\dagger B} \end{pmatrix}$$

such that

$$W^T \mathcal{M}_{D+M} W = \begin{pmatrix} \mathcal{M}_\nu & 0 \\ 0 & \mathcal{M}_\nu^{\text{heavy}} \end{pmatrix}$$

Square root understood as $\sqrt{1-x} = 1 - \frac{1}{2}\,x - \frac{1}{8}\,x^2 - \cdots$

B is function of M_D, M_R, expansion in m_D/m_R

One can show: $B = B_1 + B_3 + B_5 + \cdots$, B_n of order $(m_D/m_R)^n$

Recursive solution $\boxed{B_1 = \left(M_R^{-1} M_D\right)^\dagger}$ Schechter, Valle 1982

$$W \simeq \begin{pmatrix} 1 - \frac{1}{2}\, B_1 B_1^\dagger & B_1 \\ -B_1^\dagger & 1 - \frac{1}{2}\, B_1^\dagger B_1 \end{pmatrix}$$

Mass matrix of light νs: $\boxed{\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D}$

Mass matrix of heavy νs: $\boxed{\mathcal{M}_\nu^{\text{heavy}} = M_R}$

Corrections to $\mathcal{M}_\nu$, $\mathcal{M}_\nu^{\text{heavy}}$ suppressed by $(m_D/m_R)^2$

Diagonalization: $(U_R^\ell)^\dagger M_\ell U_L^\ell = \hat{m}_\ell$, $V^T \mathcal{M}_\nu V = \hat{m}$

Mixing matrix: $\boxed{U_M = (U_L^\ell)^\dagger V}$

3 sources for ν mixing: M_ℓ, M_D, M_R

Seesaw mechanism rich playground for model building!!!

Scales: choose $m_\nu \sim \sqrt{\Delta m^2_{\mathrm{atm}}} \sim 0.05$ eV

$m_D \sim m_{\mu,\tau} \Rightarrow m_R \sim 10^8 \div 10^{11}$ GeV

Could seesaw scale m_R be identical with the GUT scale

$M_{\mathrm{GUT}} \simeq 2 \times 10^{16}$ GeV?

$m_D \lesssim v$ with ew. scale $v = 174$ GeV $\Rightarrow$

$m_\nu \sim v^2/M_{\mathrm{GUT}} \sim 1.5 \times 10^{-3}$ eV $\Rightarrow$ too small!

MSSM gauge coupling unification at M_{GUT}, no intermediate scale!

Rules out minimal SUSY $SO(10)$ GUT $\rightarrow$ hot research topic

(b) Additional scalar multiplets

Leptonic SM multiplets: (irreps denoted by weak isopsin)

	$SU(2)$	$\times$ $U(1)$	
D_L	$\underline{\frac{1}{2}}$	$Y = -1$	left-handed doublets
ℓ_R	$\underline{0}$	$Y = -2$	right-handed singlets

Leptonic bilinears (Konetschny, Kummer 1977 [3]):

$\bar{D}_L \otimes \ell_R$	$\underline{\frac{1}{2}} \otimes \underline{0} = \underline{\frac{1}{2}}$	$Y = -1$	ϕ	doublet $Y = +1$
$D_L \otimes D_L$	$\underline{\frac{1}{2}} \otimes \underline{\frac{1}{2}} = \underline{0} \oplus \underline{1}$	$Y = -2$	η^+	singlet $Y = +2$
			Δ	triplet $Y = +2$
$\ell_R \otimes \ell_R$	$\underline{0} \otimes \underline{0} = \underline{0}$	$Y = -4$	k^{++}	singlet $Y = +4$

Complete SM list!

Add $\nu_R \otimes \nu_R$: $\underline{0} \otimes \underline{0} = \underline{0}$, $Y = 0 \Rightarrow$ singlet χ (real or complex)

Zee model: SM with $2\phi + \eta^+$ Zee 1980

$$\mathcal{L} = \cdots + f_{\alpha\beta} D^T_{\alpha L} C^{-1} i\tau_2 D_{\beta L}\, \eta^+ - \mu\, \phi_1^\dagger \tilde{\phi}_2 \eta^+ + \text{H.c.}$$

Note: $f_{\alpha\beta} = -f_{\beta\alpha}$

Lepton number:

	D_L	ℓ_R	$\phi_{1,2}$	η^+
L	1	1	0	-2

ν masses $\Rightarrow$ L must be broken

L explicitly broken by μ-term $\longrightarrow$ two Higgs doublets!

(if $\phi_1 = \phi_2 = \phi \Rightarrow \phi^\dagger \tilde{\phi} \equiv 0$)

Restricted Zee model: Wolfenstein 1980

Only ϕ_1 couples to leptons via symmetry

$D_L \to iD_L$, $\ell_R \to i\ell_R$, $\phi_1 \to \phi_1$, $\phi_2 \to -\phi_2$, $\eta^+ \to -\eta^+$

$\Rightarrow$ 1-loop ν masses: $\mathcal{M}_\nu \propto \Big((m_\alpha^2 - m_\beta^2) f_{\alpha\beta} \Big)$

Most general Majorana mass matrix with zeros on the diagonal $\to$ 3-parameter mass matrix! (after removal of unphysical phases)

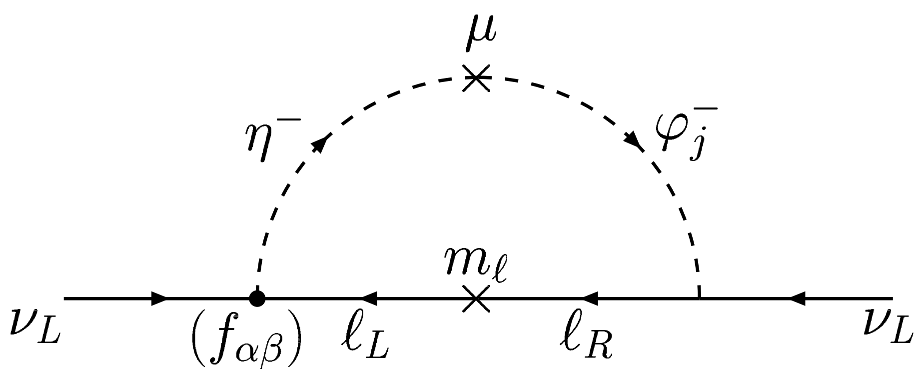
Results:

- ∗ Solar mixing maximal! (Jarlskog et al.; Frampton, Glashow)
- ∗ Fine-tuning $m_\tau^2 |f_{e\tau}| \simeq m_\mu^2 |f_{e\mu}|$
- ∗ Small ν masses because $|f_{\alpha\beta}| \lesssim 10^{-4}$

Note: If $(\mathcal{M}_\nu)_{ee} \neq 0 \rightarrow$ interesting viable mass matrix

Straightforward remedy for non-maximal solar mixing:

Couple both $\phi_{1,2}$: non-maximal solar mixing admitted



1-loop Majorana ν masses in the Zee model

Note: Majorana mass term corresponds to transition $\nu_L \rightarrow (\nu_L)^c$
because $\nu_L^T C^{-1} \nu_L = -\overline{(\nu_L)^c}\, \nu_L$

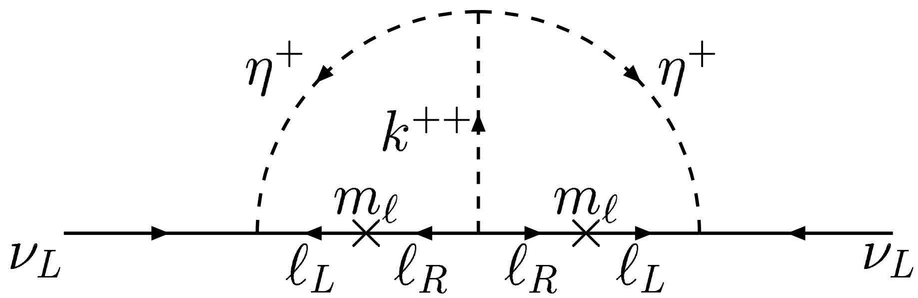
Zee-Babu model: SM + η^+ + k^{++} Babu 1988

$$\begin{aligned}\mathcal{L} \; = \; & \cdots + f_{\alpha\beta} D_{\alpha L}^T C^{-1} i\tau_2 D_{\beta L}\, \eta^+ \\ & + h_{\alpha\beta} \ell_{\alpha R}^T C^{-1} \ell_{\beta R}\, k^{++} - \mu\, \eta^- \eta^- k^{++} + \text{H.c.}\end{aligned}$$

Note: $h_{\alpha\beta} = h_{\beta\alpha}$ $\quad L(k^{++}) = -2 \;\; L$ explicitly broken by μ-term!

2-loop ν masses: $\mathcal{M}_\nu \propto \tilde{f} \hat{m}_\ell \tilde{h}^* \hat{m}_\ell \tilde{f}$

with $\tilde{f} = (f_{\alpha\beta})$, $\tilde{h} = (h_{\alpha\beta})$, $\hat{m}_\ell = \text{diag}\,(m_e, m_\mu, m_\tau)$



Properties of the model:

- $\tilde{f}$ antisymm. $\Rightarrow$ lightest ν mass $m_0 = 0$
- ν mass hierarchy: fine-tuning (Babu, Macesanu)
  $|h_{\mu\mu}| : |h_{\mu\tau}| : |h_{\tau\tau}| \simeq 1 : (m_\mu/m_\tau) : (m_\mu/m_\tau)^2$
- Scalar masses in TeV range
- $|f_{\alpha\beta}|$, $|h_{\alpha\beta}| \lesssim 0.1 \Rightarrow \nu$ masses naturally small!
- Rare decays $\tau \to 3\mu$, $\mu \to e\gamma$ within reach of forthcoming experiments

Radiative ν mass generation: Hierarchy of charged lepton masses goes into neutrino masses $\Rightarrow$ finetuning of Yukawa couplings

Triplet model: SM + Δ

$$\begin{aligned} \mathcal{L} &= \cdots + \tfrac{1}{2} g_{\alpha\beta} D_{L\alpha}^T C^{-1} i\tau_2 \Delta D_{L\beta} + \text{h.c.} \\ &\quad - M^2 \text{Tr}\, \Delta^\dagger \Delta - \left(\mu\, \phi^\dagger \Delta \tilde{\phi} + \text{h.c.} \right) - \cdots \end{aligned}$$

$$\Delta = \begin{pmatrix} H^+ & \sqrt{2} H^{++} \\ \sqrt{2} H^0 & -H^+ \end{pmatrix}, \qquad \langle H^0 \rangle_0 = \frac{1}{\sqrt{2}} v_T$$

Note: $g_{\alpha\beta} = g_{\beta\alpha}$

$L(\Delta) = -2$, L explicitly broken by μ-term!

Tree-level ν masses: $\mathcal{M}_\nu = v_T (g_{\alpha\beta})$

Gelmini, Roncadelli (1981)

LEP data: $|v_T|/v \lesssim 0.03$ Erler, Langacker

Small ν masses by small $v_T \sim 0.1 \div 1$ eV!

$SU(2)$-invariance of triplet coupling: $U \in SU(2)$

$D_L \to U D_L$, $\Delta \to U \Delta U^\dagger$

Invariance because of $U^T i\tau_2 U = i\tau_2$

Charge-eigenfields of triplet: $\Delta = \sum_{j=1}^{3} \delta_j \tau_j$, $Y_\Delta = 2$

$\mathbb{Q} = T_3 + \frac{1}{2} Y \Rightarrow \mathbb{Q}\Delta = \frac{1}{2} [\tau_3, \Delta] + \Delta$

$$\begin{pmatrix} \delta_3 & \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 & -\delta_3 \end{pmatrix} \xrightarrow{\mathbb{Q}} \begin{pmatrix} \delta_3 & 2\,(\delta_1 - i\delta_2) \\ 0\,(\delta_1 + i\delta_2) & -\delta_3 \end{pmatrix}$$

$\Rightarrow H^0 = \frac{1}{\sqrt{2}} (\delta_1 + i\delta_2)$, $H^+ = \delta_3$, $H^{++} = \frac{1}{\sqrt{2}} (\delta_1 - i\delta_2)$

How to get small v_T: Assumption $\boxed{M,\ |\mu| \gg v}$

$\langle V \rangle_0 = M^2 v_T^* v_T + v^2 \mu v_T + v^2 \mu^* v_T^* + \cdots$

Dots contain terms of order v^4, $v_T^2 v^2$

$\frac{\partial}{\partial v_T^*} \langle V \rangle_0 = M^2 v_T + v^2 \mu^* + \mathcal{O}(v_T v^2) = 0$

$\Rightarrow \boxed{|v_T| \simeq |\mu| v^2 / M^2}$ Scalar or type II seesaw [4]

Generation of small m_ν: Triplet model needs small $v_T \Rightarrow$ new scale in scalar sector (analogy to seesaw mechanism of type I)

Seesaw type I+II are naturally obtained in $SO(10)$ GUTs

Type I+II seesaw mechanism:

$$\mathcal{M}_{D+M} = \begin{pmatrix} M_L & M_D^T \\ M_D & M_R \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} \nu_L \\ C(\bar{\nu}_R)^T \end{pmatrix}$$

$$\mathcal{M}_\nu = M_L - M_D^T M_R^{-1} M_D$$

Note: $B = B_1 + B_2 + B_3 + \cdots$ (expansion in $1/m_R$)

M_L can also appear via loop corrections in some models

Specific ν mass and mixing model

A specific mass matrix:

Majorana mass term $\mathcal{L}_{\mathrm{maj}} = \frac{1}{2}\, \nu_L^T C^{-1} \mathcal{M}_\nu \nu_L + \mathrm{H.c.}$

Basis where charged-lepton mass matrix is diagonal:

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix} \qquad x,\, y,\, z,\, w \in \mathbb{C}$$

Defining relation: μ–τ interchange symmetry

$$S \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad S \mathcal{M}_\nu S = \mathcal{M}_\nu$$

Phenomenology of the mass matrix: $V^T \mathcal{M}_\nu V = \hat{m}$

$$\begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = (z-w) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow m_3 = |z-w|$$

$$V = \mathrm{diag}\left(e^{i\varphi_e}, e^{i\varphi_\mu}, -e^{i\varphi_\mu}\right) \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \mathrm{diag}\left(e^{i\rho}, e^{i\sigma}, 1\right)$$

Angles:
$\theta_{13} = 0^\circ$
$\theta_{23} = 45^\circ$
$\theta_{12} \equiv \theta$
arbitrary

Phases:
no CKM phase!
Majorana phases
ρ, σ

Masses:
free!

$$\sin^2 2\theta_{\mathrm{atm}} = 4\,|U_{\mu 3}|^2 \left(1 - |U_{\mu 3}|^2\right) = 1$$

Remarks:

- $\mathcal{M}_\nu$ compatible with all data
- Removing 2 unphysical phases, $\mathcal{M}_\nu$ has 6 real parameters corresponding to $m_{1,2,3}$, θ_{12}, Majorana phases ρ, σ
- Form of mass matrix fixes $\theta_{13} = 0°$, $\theta_{23} = 45°$
- Third line of U has factor -1 compared to standard parameterization of U

Perturbation of $\mathcal{M}_\nu$:

$$\mathcal{M}_{\nu f} = \mathcal{M}_\nu + \delta\mathcal{M}_\nu = \begin{pmatrix} x & y\,(1+\epsilon) & y\,(1-\epsilon) \\ y\,(1+\epsilon) & z\,(1+\epsilon') & w \\ y\,(1-\epsilon) & w & z\,(1-\epsilon') \end{pmatrix}$$

with $|\epsilon| \ll 1$, $|\epsilon'| \ll 1$

Note: $S\mathcal{M}_\nu S = \mathcal{M}_\nu$, $S\,\delta\mathcal{M}_\nu S = -\delta\mathcal{M}_\nu$

$$\Rightarrow\ \delta\mathcal{M}_\nu = \begin{pmatrix} 0 & \epsilon_1 & -\epsilon_1 \\ \epsilon_1 & \epsilon_2 & 0 \\ -\epsilon_1 & 0 & -\epsilon_2 \end{pmatrix} \text{ with def. } \quad \epsilon_1 = y\epsilon,\ \epsilon_2 = z\epsilon'$$

Physical parameters: $\mathrm{Re}\,\epsilon$, $\mathrm{Re}\,\epsilon'$, $\mathrm{Im}\,(2\epsilon - \epsilon')$

Definitions: $\hat{m}_1 = e^{-2i\rho} m_1$, $\hat{m}_2 = e^{-2i\sigma} m_2$

$$
\begin{aligned}
\cos 2\theta_{23} \quad = \quad & \Bigg\{ \frac{c_{12}^2}{m_2^2 - m_3^2} \left[m_3^2 + c_{12}^2 m_2^2 + s_{12}^2 \mathrm{Re}\left(\hat{m}_1 \hat{m}_2^* + \hat{m}_1 m_3 \right) \right. \\
& \left. + \left(1 + c_{12}^2\right) \mathrm{Re}\left(\hat{m}_2 m_3 \right) \right] \\
& + \frac{s_{12}^2}{m_1^2 - m_3^2} \left[m_3^2 + s_{12}^2 m_1^2 + c_{12}^2 \mathrm{Re}\left(\hat{m}_2 \hat{m}_1^* + \hat{m}_2 m_3 \right) \right. \\
& \left. + \left(1 + s_{12}^2\right) \mathrm{Re}\left(\hat{m}_1 m_3 \right) \right] \Bigg\} \mathrm{Re}\, \epsilon' \\
& + 2 c_{12}^2 s_{12}^2 \left[\frac{m_2^2 - \mathrm{Re}\left(\hat{m}_1 \hat{m}_2^* + \hat{m}_1 m_3 - \hat{m}_2 m_3 \right)}{m_2^2 - m_3^2} \right. \\
& \left. + \frac{m_1^2 - \mathrm{Re}\left(\hat{m}_1 \hat{m}_2^* - \hat{m}_1 m_3 + \hat{m}_2 m_3 \right)}{m_1^2 - m_3^2} \right] \mathrm{Re}\, \epsilon \\
& + \frac{c_{12}^2 s_{12}^2 \left(m_1^2 - m_2^2 \right) \mathrm{Im}\left(\hat{m}_1 \hat{m}_2^* + \hat{m}_1 m_3 + \hat{m}_2^* m_3 \right)}{\left(m_3^2 - m_1^2 \right)\left(m_3^2 - m_2^2 \right)} \mathrm{Im}\left(2\epsilon - \epsilon' \right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{U_{e3}}{c_{12}s_{12}} \quad = \quad & \frac{1}{2}\left\{\frac{1}{m_2^2-m_3^2}\left[s_{12}^2\hat{m}_1\hat{m}_2^* + s_{12}^2\hat{m}_1^* m_3 + \left(1+c_{12}^2\right)\hat{m}_2^* m_3 + m_3^2 + c_{12}^2 m_2^2\right]\right. \\
& \left. + \frac{1}{m_3^2-m_1^2}\left[c_{12}^2\hat{m}_1^*\hat{m}_2 + c_{12}^2\hat{m}_2^* m_3 + \left(1+s_{12}^2\right)\hat{m}_1^* m_3 + m_3^2 + s_{12}^2 m_1^2\right]\right\} \\
& \times \mathrm{Re}\,\epsilon' \\
& + \left[\frac{s_{12}^2}{m_2^2-m_3^2}\left(m_2^2 - \hat{m}_1\hat{m}_2^* - \hat{m}_1^* m_3 + \hat{m}_2^* m_3\right)\right. \\
& \left. + \frac{c_{12}^2}{m_3^2-m_1^2}\left(m_1^2 - \hat{m}_1^*\hat{m}_2 + \hat{m}_1^* m_3 - \hat{m}_2^* m_3\right)\right]\mathrm{Re}\,\epsilon \\
& + \frac{i}{2}\left[\frac{s_{12}^2}{m_2^2-m_3^2}\left(m_2^2 - \hat{m}_1\hat{m}_2^* + \hat{m}_1^* m_3 - \hat{m}_2^* m_3\right)\right. \\
& \left. + \frac{c_{12}^2}{m_3^2-m_1^2}\left(m_1^2 - \hat{m}_1^*\hat{m}_2 - \hat{m}_1^* m_3 + \hat{m}_2^* m_3\right)\right]\mathrm{Im}\left(2\epsilon - \epsilon'\right).
\end{aligned}
$$

Furthermore: $m_{1,2,3}$, θ_{12} not changed in first order in ϵ, ϵ'

Results:

* ϵ, ϵ' induce non-zero U_{e3} and shifts θ_{23} away from 45°
* CKM phase is induced (depends on ρ, σ, $\mathrm{Im}\,(2\epsilon - \epsilon')$)
* Normal and inverted hierarchy: induced U_{e3}, $\cos 2\theta_{23}$ small
* Degenerate spectrum: in general, induced U_{e3}, $\cos 2\theta_{23}$ strongly enhanced by $m_0^2/\Delta m^2_{\mathrm{atm}}$ (common mass m_0)
  However: $\rho = \sigma = 0 \Rightarrow U_{e3}$ small

Remark: We have *not* specified mechanism for ϵ, ϵ'!
E.g., radiative corrections
Suppose at high scale S-invariant ν mass matrix $\Rightarrow$ enhancement in the degenerate case via RG running (Grimus, Kaneko, Joshipura, Lavoura, Sawanaka, Tanimoto, hep-ph/0408123)

Is there a symmetry connection between small U_{e3} and (nearly) maximal neutrino mixing?

In general no connection!

Models:

- Groups $O(2)$, D_4 $\rightarrow$ (Grimus, Lavoura) $\theta_{13} = 0°$, $\theta_{23} = 45°$
- Groups A_4 (Babu, Ma, Valle), non-standard CP (Grimus, Lavoura) $\rightarrow$ θ_{13} arbitrary, $\theta_{23} = 45°$
- Groups $\mathbb{Z}_4$ (C. Low), D_4 (Grimus, Joshipura, Kaneko, Lavoura, Tamimoto) $\rightarrow$ $\theta_{13} = 0°$, θ_{23} arbitrary

Discuss in the following example the model of [5] based on $O(2)$

The μ–τ-symmetric mass matrix and $O(2)$

The group $O(2)$: Symmetrie group in the plane

Abstract characterization:

Rotations: $g(\theta)$ with $g(\theta + 2\pi) = g(\theta)$, $g(\theta_1)\, g(\theta_2) = g(\theta_1 + \theta_2)$

Reflexion at x-axis: s with $s^2 = e$

Defining relation: $s\, g(\theta)\, s = g(-\theta)$

Irreducible representations:

Two 1-dimensional irreps:

$$\underline{1}: \; g(\theta) \to 1, \; s \to 1, \quad \underline{1}': \; g(\theta) \to 1, \; s \to -1.$$

Infinite series of 2-dimensional irreps characterized by $n \in \mathbb{N}$:

$$\underline{2}^{(n)}: \quad g(\theta) \to \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix}, \quad s \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Tensor products:

$$m > n: \qquad \boxed{\underline{2}^{(m)} \otimes \underline{2}^{(n)} = \underline{2}^{(m+n)} \oplus \underline{2}^{(m-n)}}$$

with the bases

$$\underline{2}^{(m+n)}: \quad e_1 \otimes e_1, \ e_2 \otimes e_2, \qquad \underline{2}^{(m-n)}: \quad e_1 \otimes e_2, \ e_2 \otimes e_1$$

$$m = n: \qquad \boxed{\underline{2}^{(n)} \otimes \underline{2}^{(n)} = \underline{1} \oplus \underline{1}' \oplus \underline{2}^{(2n)}}$$

with the bases

$$\underline{2}^{(2n)}: \quad e_1 \otimes e_1, \ e_2 \otimes e_2$$

$$\underline{1}: \quad \frac{1}{\sqrt{2}}\left(e_1 \otimes e_2 + e_2 \otimes e_1\right), \qquad \underline{1}': \quad \frac{1}{\sqrt{2}}\left(e_1 \otimes e_2 - e_2 \otimes e_1\right)$$

Multiplets: SM with $3\,\phi_j$ $(j=1,2,3)$ + $3\,\nu_R$

Symmetries:

- ▷ $U(1)_{L_\alpha}$ $(\alpha = e, \mu, \tau)$
  - ∗ softly broken by ν_R mass term (dim 3)!
- ▷ $\mathbb{Z}_2^{(\mathrm{tr})}: \begin{cases} D_{\mu L} \leftrightarrow D_{\tau L}\,,\ \mu_R \leftrightarrow \tau_R\,, \\ \nu_{\mu R} \leftrightarrow \nu_{\tau R}\,,\ \phi_3 \to -\phi_3\,; \end{cases}$
  - ∗ spontaneously broken by $\langle \phi_3 \rangle_0$
- ▷ $\mathbb{Z}_2^{(\mathrm{aux})}$: $\nu_{eR},\ \nu_{\mu R},\ \nu_{\tau R},\ \phi_1,\ e_R$ change sign
  - ∗ Spontaneously broken by $\langle \phi_1 \rangle_0$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{Y}} = & -y_1 \bar{D}_{eL} \nu_{eR} \tilde{\phi}_1 - y_2 \left(\bar{D}_{\mu L} \nu_{\mu R} + \bar{D}_{\tau L} \nu_{\tau R} \right) \tilde{\phi}_1 \\
& -y_3 \bar{D}_{eL} e_R \phi_1 - y_4 \left(\bar{D}_{\mu L} \mu_R + \bar{D}_{\tau L} \tau_R \right) \phi_2 \\
& -y_5 \left(\bar{D}_{\mu L} \mu_R - \bar{D}_{\tau L} \tau_R \right) \phi_3 + \mathrm{H.c.}
\end{aligned}
$$

Mass matrix: $\mathcal{L}_M = \frac{1}{2}\,\nu_R^T C^{-1} M_R^* \nu_R + \text{H.c.}$

$\mathcal{L}_M$ invariant under $\mathbb{Z}_2^{(\text{tr})}$ but not under $U(1)_{L_\alpha}$

$$M_D = \text{diag}\,(a, b, b), \quad M_R = \begin{pmatrix} m & n & n \\ n & p & q \\ n & q & p \end{pmatrix}$$

$$\mathbb{Z}_2^{(\text{tr})} \text{ invariance:} \quad S M_D S = M_D,\; S M_R S = M_R$$

$$\Rightarrow\; S \mathcal{M}_\nu S = \mathcal{M}_\nu \quad \text{with} \quad \mathcal{M}_\nu = -M_D^T M_R^{-1} M_D \;\Rightarrow$$

$$\mathcal{M}_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix} \quad \text{with} \quad x,\, y,\, z,\, w \in \mathbb{C}$$

Non-abelian part in symmetry group:

$U(1)_{L_\mu} \times U(1)_{L_\tau}$ and $\mathbb{Z}_2^{(\mathrm{tr})}$ do not commute $\rightarrow$ generate $O(2)$
$s \leftrightarrow \mathbb{Z}_2^{(\mathrm{tr})}$, $g(\theta) \leftrightarrow e^{i\theta(L_\mu - L_\tau)} \Rightarrow (D_{\mu L}, D_{\tau L})$ etc. $\in \underline{2}^{(1)}$, $\phi_3 \in \underline{1}'$

$$\boxed{U(1)_{L_e} \times U(1)_{L_\mu + L_\tau} \times O(2) \times \mathbb{Z}_2^{(\mathrm{aux})}}$$

The problem of $m_\mu \ll m_\tau$:

Expectation:

$|y_4 v_2| \sim |y_5 v_3| \Rightarrow m_\mu \sim m_\tau$

However, we need $m_\mu = |y_4 v_2 + y_5 v_3| \ll m_\tau = |y_4 v_2 - y_5 v_3|$

Technical solution of this finetuning problem:

$$\boxed{K: \quad \mu_R \to -\mu_R, \quad \phi_2 \leftrightarrow \phi_3}$$

$$\Rightarrow \boxed{y_4 = -y_5 \quad \text{and} \quad \frac{m_\mu}{m_\tau} = \left|\frac{v_2 - v_3}{v_2 + v_3}\right|}$$

Implementation of K in Higgs potential: $v_2 = v_3$ and $m_\mu = 0$

Soft breaking of K by term of dimension 2 in Higgs potential:

$m_\mu \neq 0$, $m_\mu \ll m_\tau$ technically natural

$V = V_\phi + V_{\rm soft}$ (V_ϕ K-invariant)

$V_{\rm soft} = \mu^2_{\rm soft}\left(\phi_2^\dagger\phi_2 - \phi_3^\dagger\phi_3\right)$ Unique soft K-breaking term of dim 2

$$\begin{aligned} V_\phi &= -\mu_1^2\phi_1^\dagger\phi_1 - \mu_2^2\left(\phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\right) \\ &+\lambda_1\left(\phi_1^\dagger\phi_1\right)^2 + \lambda_2\left[\left(\phi_2^\dagger\phi_2\right)^2 + \left(\phi_3^\dagger\phi_3\right)^2\right] \\ &+\lambda_3\left(\phi_1^\dagger\phi_1\right)\left(\phi_2^\dagger\phi_2 + \phi_3^\dagger\phi_3\right) + \lambda_4\left(\phi_2^\dagger\phi_2\right)\left(\phi_3^\dagger\phi_3\right) \\ &+\lambda_5\left[\left(\phi_1^\dagger\phi_2\right)\left(\phi_2^\dagger\phi_1\right) + \left(\phi_1^\dagger\phi_3\right)\left(\phi_3^\dagger\phi_1\right)\right] + \lambda_6\left(\phi_2^\dagger\phi_3\right)\left(\phi_3^\dagger\phi_2\right) \\ &+\lambda_7\left[\left(\phi_2^\dagger\phi_3\right)^2 + \left(\phi_3^\dagger\phi_2\right)^2\right] \\ &+\lambda_8\left[\left(\phi_1^\dagger\phi_2\right)^2 + \left(\phi_1^\dagger\phi_3\right)^2\right] + \lambda_8^*\left[\left(\phi_2^\dagger\phi_1\right)^2 + \left(\phi_3^\dagger\phi_1\right)^2\right] \end{aligned}$$

Note: V invariant under three independent sign changes $\phi_j \to -\phi_j$

All coupling constants real except λ_8

Ansatz: $\boxed{v_2 = u\, e^{i\alpha} \cos\sigma,\; v_3 = u\, e^{i\beta} \sin\sigma}$ $v_1 > 0,\; u > 0$ [6]

$F_\phi \equiv \langle 0\, | V_\phi |\, 0 \rangle$

$$\begin{aligned} F_\phi &= -\mu_1^2 v_1^2 - \mu_2^2 u^2 + \lambda_1 v_1^4 + \lambda_2 u^4 + (\lambda_3 + \lambda_5)\, v_1^2 u^2 \\ &\quad + \left[\tilde\lambda - 4\lambda_7 \sin^2(\alpha - \beta)\right] u^4 \cos^2\sigma \sin^2\sigma \\ &\quad + 2\, |\lambda_8|\, v_1^2 u^2 \left[\cos^2\sigma\, \cos(\epsilon + 2\alpha) + \sin^2\sigma\, \cos(\epsilon + 2\beta)\right] \end{aligned}$$

$\tilde\lambda \equiv -2\lambda_2 + \lambda_4 + \lambda_6 + 2\lambda_7,\; \epsilon \equiv \arg\lambda_8$

$\boxed{\text{If } \tilde\lambda < 0 \text{ and } \lambda_7 < 0 \Rightarrow \text{minimum of } F_\phi \text{ at } \sigma = \frac{\pi}{4},\; \alpha = \beta = \frac{\pi - \epsilon}{2}}$

$$\boxed{v_2 = v_3 = \frac{1}{\sqrt{2}}\, u\, e^{i\,(\pi - \epsilon)/2}}$$

$$\langle V_{\text{soft}} \rangle_0 = \mu_{\text{soft}}^2 u^2 \cos 2\sigma \;\Rightarrow\; \boxed{\cos 2\sigma = \frac{2\mu_{\text{soft}}^2}{\tilde\lambda u^2},\; \frac{m_\mu}{m_\tau} = \frac{|\cos 2\sigma|}{1 + \sqrt{1 - \cos^2 2\sigma}}}$$

How the model functions:

- $\mathbb{Z}_2^{(\mathrm{tr})} \Rightarrow M_{D\mu\mu} = M_{D\tau\tau}$, M_R and, therefore, $\mathcal{M}_\nu$ are μ–τ-invariant
- M_ℓ diagonal because of lepton numbers
- Consequently, atm. mixing maximal!
  At the same time, solar mixing large, $U_{e3} = 0$
- Auxiliary symmetry $\mathbb{Z}_2^{(\mathrm{aux})} \Rightarrow \phi_{2,3}$ do not couple to $\bar{\nu}_R D_L$
- $\mathbb{Z}_2^{(\mathrm{tr})}$ spontanously broken VEV of $\phi_3 \Rightarrow m_\mu \neq m_\tau$
- $\Delta m^2_\odot / \Delta m^2_{\mathrm{atm}} \sim 0.03$ reproduced by values $2 \div 3$ of ratios of elements of $M_{D,R}$!
  Reproduced by tuning

Remark: μ–τ-invariance at seesaw scale $\Rightarrow$ RG running changes $\mathcal{M}_\nu$ very little, s_{13} at most 0.1 for $m_0 \simeq 0.3$ eV

General framework: Seesaw mechanism and soft L_α breaking

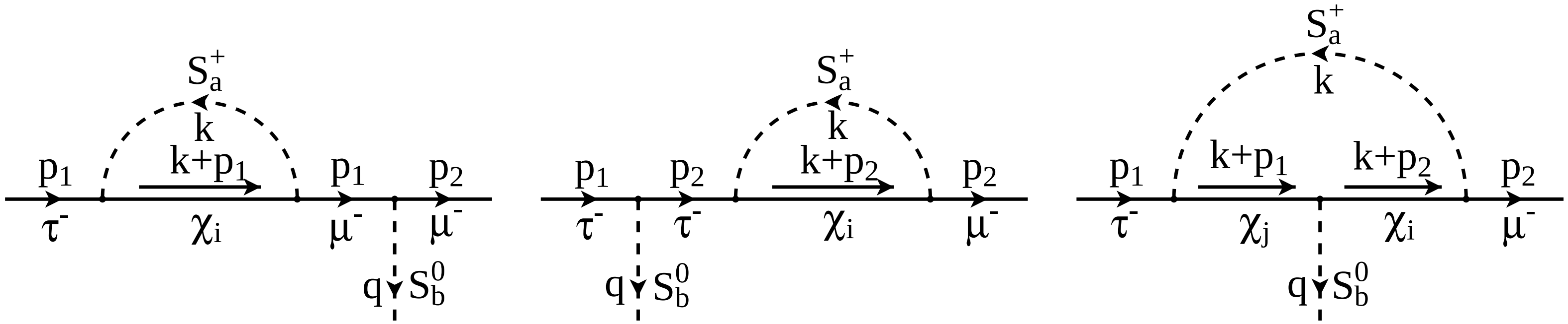
$\mathcal{M}_\nu = -M_D^T M_R^{-1} M_D$

SM with $n_H\ \phi + 3\,\nu_R$ + soft $L_{e,\mu,\tau}$ breaking by $\frac{1}{2}\,\nu_R^T C^{-1} M_R^* \nu_R$

* Renormalizeable theory!
  In particular, lepton-flavour-changing amplitudes finite!
* Soft L_α breaking by ν_R mass term at *high* scale m_R!
* Yukawa couplings diagonal $\Rightarrow M_\ell$, M_D diagonal
* M_R only source of ν mixing
* Non-decoupling in the scalar sector for $n_H > 1$ for $m_R \to \infty \Rightarrow$
  - ▷ Amplitude of, e.g., $\mu \to 3e$ constant, suppressed by product of 4 Yukawa couplings, within exp. reach?
  - ▷ Amplitudes of $\mu \to e\gamma$, $Z \to e^-\mu^+, \ldots \propto 1/m_R^2$

Flavour-changing neutral-scalar vertex:

Does not vanish for $m_R \to \infty$



Decays unsuppressed by $1/m_R$:

$\mu^- \to e^- e^+ e^-$, $\tau^- \to \mu^- e^+ e^-$, $\tau^- \to \mu^- \mu^+ \mu^-$, $\tau^- \to e^- e^+ e^-$

Suppressed decays:

$\tau^- \to \mu^- \mu^- e^+$, $\tau^- \to e^- e^- \mu^+$ (box diagrams)

Further specifiaction of mixing matrix:

Tri-bimaximal mixing (Harrison, Perkins, Scott 2002)

$$\sin^2\theta = \frac{1}{3} \Rightarrow \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\cos\theta}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$\theta = 35.3^\circ$ perfectly OK!

Family symmetry requires 3-dim irreps $\rightarrow$ very complicated models

How to incorporate presented idea(s) into $SO(10)$ GUTs?

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