Lectures on Ads/CFT Correspondence

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1 CONFORMAL SYMMETRY, CORRELATORS

In this lecture a basic review of the AdS/CFT correspondence will be given. Topics to be discussed:

1. Conformal Symmetry (and its realizations)
2. Correlators, GKP-W, S-matrix
3. Ward Identities

References

[1] V.Balasubramanian, Steven B. Giddings, Albion E. Lawrence, "What do CFTs tell us about Anti-de Sitter space-times?" *JHEP* **9903**, 001(1999) [arXiv:hep-th/9902052].

[2] M. Gunaydin, D. Minic, M. Zagermann, "4-D doubleton conformal theories, CPT and IIB string on AdS(5) x S-5". e-Print Archive: hep-th/9806042

1.1 Some basic representation theory

Following: M. Gunaydin, D. Minic and M. Zagermann

1.2 $SO(4,2)$

The five-dimensional AdS group and the four-dimensional conformal group are isomorphic to $S0(4,2)(A=0,\ \cdots,3,5,6)$. The algebra reads:

$$[M_{AB},M_{CD}]=i(\eta_{BC}M_{AD}+3\text{ more})\,,$$

with $\eta_{AB}=\text{diag}(-1,+1,+1,+1,+1,-1)$. The maximal compact subgroup of $SO(4,2)$, (its spin extension) $SU(2,2)$, is $L^0 = SU(2)_L\times SU(2)_R\times U(1)_E$,and is given by the generators $(i=1,2,3)$:

$$\begin{aligned} SU(2)_L &: \quad L_i=\frac{1}{2}(M_{5i}+\frac{1}{2}\epsilon_{ijk}M_{jk})\,,\\ SU(2)_R &: \quad R_i-\frac{1}{2}(-M_{5i}+\frac{1}{2}\epsilon_{ijk}M_{jk}\,,\\ U(1)_E &: \quad E=-M_{06}\,. \end{aligned}$$

The other generators of $SO(4,2)$ are split into L^+ (raising)operators and L^- of (lowering) operators, such that $[E,L^{\pm}]=\pm L^{\pm},[L^+,L^+]=0=$

$[L^-, L^-]$ and $[L^+, L^-] = L^0$. Unitary positive energy representations of $S0(4,2)$ come from weight spaces $D(j_L, j_R; E)$ formed by acting with L^+ on ground states of lowest weight states, $|j_L, j_R; E\rangle$ which are annihilated by L^- and form a representation of L^0 labeled by $(j_L, j_R; E)$. They can be generated by the oscillator method where one represents

$$M_{AB} = \frac{1}{2}\bar{y}\Sigma_{ABy}\,,$$

with the four-component $SO(4,1)$ Dirac spinor y_α and its conjugate $\bar{y}^\alpha$ obeying

$$[y_\alpha, \bar{y}^\beta]_* = 2\delta_\alpha^\beta\,,$$

Here

$$\Sigma_{ab} = -\frac{i}{2}\Gamma_{ab}\ ,\ \Sigma_{a6} = -\frac{i}{2}\Gamma_a\,.$$

correspond to a standard Dirac matrices:

$$\Gamma^0 = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\ \Gamma^i = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\ ,\ \Gamma^5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,.$$

This splits y_α into the following $U(2)$ invariant oscillators $(I = 1,2, P = 1,2)$:

$$y_\alpha = \sqrt{2}\begin{pmatrix} a^I \\ b_P \end{pmatrix}\ ,\ \bar{y}^\alpha = \sqrt{2}(-a_I, b^P)\ ,\ a^I = (a_I)^\dagger\ ,\ b^P = (b_P)^\dagger\,.$$

The compact $SO(4,2)$ generators are represented as :

$$\begin{aligned} L_i &= \frac{1}{2}(\sigma^i)_I{}^J L_J{}^I\ ,\ R_i = \frac{1}{2}(\bar{\sigma}^i)^P{}_Q R^Q{}_P\,, \\ E &= \frac{1}{2}(a^I a_I + b^P b_P) = \frac{1}{2}(N_a + N_b + 2)\,. \end{aligned}$$

The remaining $SO(4,2)$ generators are

$$L_{IP} = a_I b_P\ ,\ L^{IP} = a^I b^P\,,$$

and satisfy the algebra

$$[L_{IP}, L^{JQ}]_* = \delta_I^J R^Q{}_P + \delta_P^Q L^J{}_I + \delta_I^J\delta_P^Q E\,.$$

The lowest weight states of the oscillator Hilbert space are given by

$$|(j,0;j+1)\rangle = a^{I_1}\cdots a^{I_{2j}}|0\rangle,\ ,\ |(0,j;j+1)\rangle = b^{P_1}\cdots b^{P_{2j}}|0\rangle\ ,\ j = 0, \frac{1}{2}, 1\cdots$$

The weight space $D(j,0;j+1)$ and $D(0,j;j+1)$ are known as the $SO(4,2)$ doubleton representations. A tensor product is obtained from an oscillator algebra with N copies $(r,s=1,\cdots,N)$ and

$$M_{AB} = \sum_r = M_{AB}(r)\,.$$

The lowest energy states listed with increasing energy are then

$$\begin{aligned}
|(0,0;2)\rangle &= |0\rangle\,,\\
|(\frac{1}{2},\frac{1}{2};3)\rangle &= \Big(a^I(1)b^P(1) - a^I(2)b^P(2)\Big)\,|0\rangle\,,\\
|(1,1;4)\rangle &= \Big(a^I(1)a^J(1)b^P(1)b^Q(1) - 4a^{(I}(1)a^{J)}(2)b^{(P}(1)b^{Q)}(2)\\
&\quad +a^I(2)a^J(2)b^P(2)b^Q(2)\Big)\,|0\rangle\,,
\end{aligned}$$

1.3 The $SO(6)R$ Symmetry Group

Representations of the $SO(6)$ group can be constructed in terms of eight fermionic oscillators or four complex oscillators transforming in the spinor representation of $SO(6)$ as

$$\xi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -\beta^{\dot{1}\dagger} \\ -\beta^{\dot{2}\dagger} \end{pmatrix}$$

The oscillators now obey anti-commutation relations

$$\{\alpha^{\tau\dagger},\alpha_\upsilon\} = \delta^\tau_\upsilon,\;\; \{\beta^{\dot{\tau}\dagger},\beta_\upsilon\} = \delta^{\dot{\tau}}_{\dot{\upsilon}},\;\; \{\alpha_\tau,\beta_{\dot{\tau}}\} = 0,\;\; \{\alpha^{\tau\dagger},\beta_{\dot{\tau}}\} = 0$$

with the Fock space vacuum

$$\alpha_\tau|0\rangle = \beta_{\dot{\tau}}|0\rangle = 0.$$

The $SO(6)$ generators are given by

$$M_{IJ} = \xi^\dagger m_{IJ\xi}\,,$$

where m_{IJ} are four dimensional representation of generators of $SO(6)$. The graded decomposition of $SO(6)$ is $L^+ \oplus L^- \oplus L^0$ with

$$L^+ = \{\alpha^{\tau\dagger}\beta^{\tau\dagger}\}, \quad L^- = \{\alpha_\tau\beta_{\dot{\tau}}$$

$$L^0 = SU(2)_{L'} \times SU(2)_{R'} \times U(1)_J,$$

$$SU(2)_{L'} : \alpha^\dagger \vec{\sigma} \alpha, \quad SU(2)_{R'} : \quad \beta^\dagger \vec{\sigma} \beta\,,$$

$$U(1)_J : J = \xi^\dagger M_{56\xi} = \frac{1}{2}(2 - N_\alpha - N_\beta)$$

with

$$N_\alpha = \alpha^{1\dagger}\alpha_1 + \alpha^{2\dagger}\alpha_2, \qquad N_\beta = \beta^{\dot{1}\dagger}\beta_{\dot{1}} + \beta^\dagger\beta_{\dot{2}}\,.$$

The Fock space of oscillators for $SO(6)$ is constrained by

$$B = N_\alpha - N_\beta, = 0$$

One has the states

$$\begin{array}{rcl} |0\rangle & \leftrightarrow & Z \\ \alpha^{\tau\dagger}\beta^{\dot{\tau}\dagger}|0\rangle & \leftrightarrow & \phi_i; i = 1, \cdots 4 \\ \alpha^{1\dagger}\alpha^{2\dagger}\beta^{\dot{1}\dagger}\beta^{\dot{2}\dagger}|0\rangle & \leftrightarrow & \bar{Z} \end{array}$$

1.3.1 The supersymmetry generators

$\mathcal{N} = 4$ Yang-Mills admits 16 Poincaré supersymmetry generators and 16 superconformal supersymmetries. Their realization in terms of the oscillator construction is given by

$$Q^+ = \begin{cases} a^\dagger_{\dot{\gamma}}\alpha_\tau \\ b^\dagger_\gamma \beta_{\dot{\tau}} \end{cases}$$

$$Q^- = \begin{cases} a^\dagger_\gamma \beta^\dagger_{\dot{\tau}} \\ b^\dagger_\gamma \alpha^{\tau\dagger} \end{cases}$$

$$S^- = \begin{cases} a_{\dot{\tau}}\alpha^{\tau\dagger} \\ b^\gamma \beta^{\dot{\tau}}\beta^{\dot{\tau}\dagger} \end{cases}$$

$$S^+ = \begin{cases} a_{\dot{\gamma}}\beta_{\dot{\tau}} \\ b^\gamma \alpha_\tau \end{cases}$$

2 MATRIX MODEL MAPS, BPS GEOMETRIES

Considerable insight into the correspondence can be gained from the viewpoint of reduced (matrix) models where large N matrix model techniques become applicable. Here a restriction to BPS states can lead to great simplifications in gauge theory and in gravity.

In Yang-Mills theory one has a reduction to a matrix model which is simple to describe. One starts with $N=4$ super Yang-Mills theory defined on $R \times S_3$ where R stands for the time variable t. The theory contains the gauge field $A_\mu(t,\hat{x})$, scalar Higgs fields $\phi_i(t,\hat{x}), i = 1 \cdots 6$ and fermionic partners. A Kaluza-Klein expansion in terms of spherical harmonics on S_3 leads to matrix model reduction . In the simplest scheme on keeps only the zero mode degrees of freedom resulting in a matrix model. For the Higgs scalars: $\phi_i(t,\hat{x}) \to \Phi_i(t)$ we have

$$L = \mathrm{Tr}\left(\frac{1}{2}\dot{\Phi}_i^2 - \frac{1}{2R^2}\,\Phi_i^2 + \frac{1}{4}\,[\Phi_i,\Phi_j]^2\right)$$

The holomorphic notation introduces $Z_i = \Phi_i + i\Phi_{i+3}$ and its complex conjugate $\bar{Z}_i$ representing an $SU(3)$ triplet. The matrix reduction, in addition, also contains analogue matrices coming from the gauge field A_μ and also fermion.

2.1 $\frac{1}{2}$ BPS

The simplest sector consists of BPS states with 1/2 supersymmetry. In this case a very complete mapping was established in terms of the fermion (droplet) picture of matrix eigenvalues [1]. This map is essentially the same as the old $c=1$ matrix model map which successfully mapped 1d matrix quantum mechanics into 2D string . In Yang-Mills theory restriction to $\frac{1}{2}$ BPS configurations corresponds to considering a subset of correlators involving only the chiral primary operators of the general form

$$\mathrm{Tr}Z^{k_1}\,\mathrm{Tr}Z^{k_2}\cdots\mathrm{Tr}Z^{k_n}.$$

Consequently this sector is represented by the (holomorphic) sector corresponding to one of the (complex) Higgs matrices. One has the complex matrix model

$$H = \mathrm{Tr}\left(\dot{Z}^\dagger\dot{Z} + Z^\dagger Z\right)$$

and considers a reduction analogous to the Hall effect. In Hilbert space, one introduces the operators $A^\dagger = \frac{1}{2}(Z + i\Pi)$, $B = \frac{1}{2}(Z - i\Pi)$, in terms of which, the Hamiltonian and the $U(1)$ charge are simply

$$\begin{aligned} H &= \mathrm{Tr}(A^\dagger A + B^\dagger B) \\ J &= \mathrm{Tr}(A^\dagger A - B^\dagger B) \end{aligned}$$

In the sequence of eigenstates given by

$$\begin{aligned} \mathrm{Tr}\left((A^\dagger)^n\right)|0\rangle \quad & E = J = n \\ \mathrm{Tr}\left((B^\dagger)^n\right)|0\rangle \quad & E = -J = n \\ \mathrm{Tr}\left((A^\dagger)^n\right)\left((B^\dagger)^m\right)|0\rangle, \quad & E = n + m \quad , J = n - m \end{aligned}$$

a reduction to $\frac{1}{2}$ BPS configurations corresponds to a restriction to a sub sector given by A oscillators. It is useful to diagonalize the matrix A obtaining a fermion oscillator model where we have general fermionic eigenstates given by Slater determinants. After dividing by the Vandermonde determinant these relate to characters of representations of $SU(N)$

$$\psi_{B;l_1,l_2 \cdots l_N}(\lambda_1, \lambda_2, \cdots, \lambda_n) = e^{-\Sigma_i \bar{\lambda}_i \lambda_i} \chi_{l_1,l_2,\cdots,l_N}(\lambda_1, \lambda_2, \cdots, \lambda_N)$$

where $\chi_{l_1,l_2,\cdots l_N}$ corresponds to a representation whose Young tableaux consists of l_1 boxes in the first row, l_2 boxes in the second one, etc. Of special interest are the particular states corresponding to representations that contains 1 row of l boxes and another sequence states that corresponds to a representation that contains 1 column of l boxes. In the fermionic picture [**?**], the first set of states represents particles and the second corresponds to holes. These sets of states were explained to correspond to giant gravitons in AdS and on the sphere respectively in [1]. The collective field description of the system is given in terms of the density *ie* the moments $\phi_n = \sum_{j=0}^{N} \lambda_j^n$. In terms of these the states are given by nonlinear Schurr polynomials . Their dynamics is given by a 1+1 dimensional collective Hamiltonian with cubic interaction:

$$H = \hbar n A_n^\dagger A_n + \sqrt{m_1 n_2 n_3} A_{n_1}^\dagger A_{n_2} A_{n_3}$$

In conformal field notation this reads

$$L = \frac{1}{2\pi} \int dt \int dx \left[y_+ \partial_x^{-1} \dot{y}_- - \left((y_+^3 - y_-^3) + x^2 (y_+ - y_-) \right) \right]$$

with left and right moving conformal field. These can be seen to parametrize the boundary of the fermion droplet $y_{\pm}(x,t)$. If one parametrize the boundary in terms of radial coordinates, the Lagrangian becomes quadratic, a manifestation of its integrability.

Lin, Lunin and Maldacena[2] have identified a nonlinear Ansatz for 10 dimensional SUGRA which exactly reduces to the above, bosonic hamiltonian of 1d fermions. To summarize the main features of their Ansatz, one has for the 10 dimensional metric:

$$\begin{aligned} ds^2 &= -h^{-2}(dt + V_i dx^i)^2 + h^2(dy^2 + dx^i dx^i) + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \\ h^{-2} &= 2y\cosh G\,, \\ z &= \frac{1}{2}\tanh G \end{aligned}$$

and a corresponding Ansatz for the gauge fields. The only unknown function z is shown to obey the Laplace equation:

$$\partial_i\partial_i z + y\partial_y\left(\frac{\partial_y z}{y}\right) = 0$$

Remarkably, the flux and the energy of this general configuration were shown by LLM to take the form of the bosonized free fermion droplet

$$\begin{aligned} N &= \frac{1}{4\pi^2 l_P^2}\int dx_1 \int dx_2 \left(u(t,x_1,x_2) + \frac{1}{2}\right) \\ E &\Rightarrow \frac{1}{4\pi\hbar^2}\int dx_1 \int dx_2 \left(x_1^2 + x_2^2\right)\left(u\left(t,x_1,x_2\right) + \frac{1}{2}\right) \end{aligned}$$

It is useful to discuss the map [3] between $\frac{1}{2}$ BPS SUGRA states and eigenstates of the one matrix model at the linearized level. In SUGRA, one has the highest weight states (in both AdS_5 and S_5) and the set of wavefunctions

$$\psi_j(t,q,\Theta,\phi) \sim \left(\frac{\cos\Theta}{\cosh\rho}e^{i\phi}\right)^j e^{-ijt}$$

This is to be compared with the one dimensional set of matrix theory wavefunctions given simply as

$$\tilde{\psi}_n(t,\tau) \sim e^{-int}\, e^{in\tau}$$

One can describe a $1-1$ correspondence between these (linearized) wavefunction through a nonlocal transform

$$\psi_j\left(t, a = \frac{\cos\Theta}{\cosh\rho}, \phi\right) = e^{ijt}\int K(a|\tau)e^{ijt}d\tau$$

with the kernel

$$K(a|\tau) = \frac{1 - 4a^2 + 4a^3\cos(\tau - \phi)}{[1 + a^2 - 2a\cos(\tau - \phi)]^2}$$

This kernel summarizes the nontrivial map between SUGRA space-time and the matrix coordinates. Generalizations of it will be described.

2.2 AdS Bulk

The most interesting set of observables and the most relevant part of the map [3] concerns the bulk of AdS space-time. We have already seen in the study of holomorphic observables the need for understanding the expectation values involving for example $\bar{Z}Z + \bar{\Phi}\Phi + \bar{\Psi}\Psi$. Indeed in the matrix model realization of conformal symmetry the square of the Higgs is nothing but the generator of special conformal transformations. This implies its direct relevance for reconstructing the radial (bulk) coordinate of AdS.

The group theoretic argument consists in the following. Consider the simplest case of the complex matrix model with $Z = A + B^\dagger$, $\bar{Z} = A^\dagger + B$. We have that

$$L_+ = \mathrm{Tr}(A^\dagger B^\dagger)\,,\; L_- = \mathrm{Tr}(AB)$$

generate the $SL_2(R)$ conformal group. Consequently in order to go away from the highest weight states one has insertions of the matrix B and for studying the bulk coordinate of AdS, one has to address the full multi matrix problem. This was accomplished for the two matrix cases of Z and $\bar{Z}$.

On the matrix theory side, wavefunctions were defined

$$\tilde{\psi}_s(x) = \sum^{1} \mathrm{Tr}\left(\delta(X - x)B^{\dagger_s}\right)$$

where $X = A + A^\dagger$ is a hermitian matrix whose dynamics govern the dynamics of $\frac{1}{2}BPS$ configuration. The second matrix (creation) operator $B^\dagger$ is treated as "impurity" with a symmetrization of nonconnecting factors in the above refined observable. It was shown that this wavefunction obeys an integral (Marchesini-Onofri) equation, which was constructed and the integral equation can be reduced to

$$\left(i|\frac{\partial}{\partial\alpha}| + s\right)\tilde{\psi}(\alpha) = -\omega\tilde{\psi}(\alpha)$$

with a solution

$$\tilde{\psi}_{j,s}(\alpha) = \beta^s\,\frac{\sin(j + 2s)\alpha}{\sin\alpha}$$

This matrix theory eigenfunctions span a two-dimensional space corresponding to the two matrices $Z = \phi_1 + i\phi_2$ of the model under consideration. In parallel to the $\frac{1}{2}BPS$ case, one can construct a map to a two dimensional subspace of $Ads \times S$ wavefunctions. The wavefunctions obey the well known Laplacian equations associated with $AdS \times S$. One identifies a two integer subset (with highest weight in S). The explicite form of the nontrivial AdS wavefunction is given in terms hypergeometric function

$$\psi_{j,s}\left(\rho;\phi,\theta\right) = \left(\cos\theta e^{i\phi}\right)^j \times \rho F\left(1-j-s,-s;1;-\sinh^2\rho\right)$$

Our map to matrix model wavefunctions follows from an integral representation which can be brought to the form

$$\psi_{jn} = \int_c \frac{1}{2\pi i z}\left[e^{i\phi}u\right]^{j+2s}\left[e^{-2i\phi}v\right]^s$$

where u and v are specific functions of radial AdS coordinate ρ. In this way we have a kernel

$$\psi_{js}(\rho;\phi,\theta) = \frac{1}{4\pi^2}\int dg \int dt K(\rho,\phi,\Theta|\sigma,\tau)e^{i(j+2s)\tau+s\sigma)}$$

defining a **map** between $AdS \times S$ and the matrix model wavefunctions. This map was based on two matrices $Z, \bar{Z}$ but an analogue generalization would apply to full six Higgs matrices.

We emphasize that this mapping to the bulk of AdS space-time involved nontrival properties of matrices and equations that they obey, and cannot be accomplished by diagonal (or "commuting" matrices). Furthermore these mappings are not holographic and involve explicitely the extra dimension.

References

[1] S. Corley, A. Jevicki and S. Ramgoolam, "Exact Correlators of Giant Gravitons from Dual $N = 4SYM$ Theory," *Adv. Theor. Math. Phys.* **5**, 809 (2002) [arXiv:hep-th/0111222].

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[3] A. Donos, A. Jevicki and J. Rodrigues, "Matrix Model Maps in AdS/CFT," *Phys. Rev.* **D72**: 125009 (2005)[arXiv:hep-th/0507124.

3 LATTICE STRINGS, HOLOMORPHIC SECTOR

Generalizations which include the full set of (stringy) observables are highly nontrivial. Success was achieved in the large J limit and for a particular subset involving holomorphic observables of the general form

$$\mathrm{Tr}\left(Z_1^{n_1} Z_2^{n_2} Z_3^{n_3} Z_1^{N_1'} Z_2^{n_2^1} \cdots \right).$$

The map can be built (based on the BMN correspondence[1, 2]) and with the methods of collective field theory which (for SUGRA states) do not in principle require the large j limit. We describe in short the essential details of the collective field construction which for these holomorphic observables was given in [3]. Denoting $Z_1 = Z$, $Z_2 = \Phi$, $Z_3 = \psi$, one can for illustrative purposes concentrate on the observables (Loops) built out of two matrices:

$$O_n^j = \sum \left(\phi^n Z^j\right)$$

Here for correspondence with SUGRA one has a symmetrization denoted by $\sum$ (since the different matrices do not commute). Hamiltonian action through, $\frac{\partial}{\partial Z}\frac{\partial}{\partial Z}$,involves also the conjugate (anti-holomorphic) loops, in particular, one has joining of holomorphic and anti-holomorphic loops

$$\left(\bar{O}_{n_i}^{j_i}\, O_{n_2}^{j_2}\right)$$

This joining then leads to mixed traces, namely ones involving $Z's$ and $\bar{Z}'s$. This large set of nontrivial loops needs to be projected to the sector under consideration, namely purely holomorphic loops. One also has the fact that the mixed loops can have a nonzero expectation value. One has the ground state wavefunction

$$\psi_0 \sim e^{-\mathrm{Tr}(\bar{Z}Z)-\mathrm{Tr}(\bar{\Phi}\Phi)}$$

in terms of which one is able to evaluate the needed expectation values

$$\langle\left(\bar{O}_{n_i}^{j_i}\, O_{n_2}^{j_2}\right)\rangle_0$$

for observables involving an equal number of $Z'X$ and $\bar{Z}'s$ (likewise for $\phi' s$). This nontrivial problem and the projection to holomorphic variables was successfully solved in [3] and a collective hamiltonian describing the symmetrized holomorphic traces was derived up to cubic level. It takes the form

$$H = (j+n)A^{\dagger}_{jn}\, A_{jn} + \frac{1}{N}\left[(j_1+n_1)-(j_2+n_2)-(j_3+n_3)\right]\cdot v_{123}$$
$$\left(A^{\dagger}_{j_1 n_1} A_{j_2 n_2}\, A_{j_3 n_3} + h.c.\right)$$

with v_{123} in agreement with corresponding interactions of AdS supergravity.

For string-like states the g^2_{YM} effects become relevant. We begin with the dimensionally reduced Yang-Mills system (matrix quantum mechanics) and discuss simple, illustrative examples below. The reduced Hamiltonian reads

$$H = \sum_{i=1}^{6} \mathrm{Tr}\left(-\frac{\partial^2}{\partial \phi_i^2} + \phi_i^2\right) - g^2_{YM} \sum_{i<j} \mathrm{Tr}\left([\phi_i, \phi_j]^2\right), \qquad i,j = 1, ..., 6.$$

with

$$\phi = \phi_1 + i\phi_2, \qquad \psi = \phi_3 + i\phi_4, \quad Z = \phi_5 + i\phi_6,$$

The interaction term in the above Hamiltonian is equivalently written as

$$\begin{aligned} H_1 = g^2_{YM}\Big(\quad & +\frac{1}{2}[\phi,\bar\phi][\psi,\bar\psi] + \frac{1}{2}[\phi,\bar\phi][Z,\bar Z] \\ & +\frac{1}{2}[\psi,\bar\psi][Z,\bar Z] \\ & -[\bar\phi,\bar Z][\phi,Z] - [\bar\psi,\bar Z][\psi,Z] - [\bar\phi,\bar\psi][\phi,\psi]\Big), \end{aligned}$$

showing the usual split into D and F terms respectively. Having in mind the passage to the infinite momentum frame, we work in a coherent state basis and project

$$\bar Z \to A^{\dagger} + B \to A^{\dagger}, \qquad Z = A + B^{\dagger} \to \frac{\partial}{\partial A^{\dagger}}\,.$$

States with $B^{\dagger}$ quanta correspond, in the pp-wave string field theory, to modes with $p^+ < 0$ and as such decouple.

One also projects the complex impurity fields ψ and ϕ

$$\begin{aligned} \bar\phi \quad & \to b^{\dagger} + d \to b^{\dagger}, \qquad \phi \to b + d^{\dagger} \to \frac{\partial}{\partial b^{\dagger}}, \\ \bar\psi \quad & \to c^{\dagger} + e \to c^{\dagger}, \qquad \psi \to c + e^{\dagger} \to \frac{\partial}{\partial c^{\dagger}} \end{aligned}$$

This last projection ensures the subspace near to chiral primary operators a truncation which could be justified by appealing to supersymmetry. On a more pedestrian level the truncation is justified by the fact that it reproduces the impurity number conserving amplitudes in agreement with pp wave light cone string field theory. Furthermore, the correct string mass spectrum (i.e. anomalous dimensions) is obtained to order $1/N^2$.
We therefore arrive at the Hamiltonian

$$\hat{H} = -g_{YM}^2\Big([b^\dagger, A^\dagger][\frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger}] + [c^\dagger, A^\dagger][\frac{\partial}{\partial c^\dagger}, \frac{\partial}{\partial A^\dagger}] + [b^\dagger, c^\dagger][\frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial c^\dagger}]\Big)$$

In the earlier discussion we have already demonstrated agreement for a class of sugra states . A three index family given by the loop variables

$$O^J_{n,m} = \sum \mathrm{Tr}(\phi^n \psi^m Z^J).$$

with the sum over all possible permutations of the ϕ and ψ fields, (corresponding chiral primaries). Methods of collective field theory lead to the following interacting cubic hamiltonian for these observables

$$\begin{aligned} H &= 2\mu\delta_{J_1,J_2+J_3}\delta_{n_1,n_2+n_3}\delta_{m_1,m_2+m_3}\frac{\sqrt{J_1J_2J_3}}{N}\sqrt{\frac{n_1!}{n_2!n_3!}}\sqrt{\frac{m_1!}{m_2!m_3!}} \\ &\times\Big(\frac{J_2}{J_1}\Big)^{\frac{n_2+m_2}{2}}\Big(\frac{J_3}{J_1}\Big)^{\frac{n_3+m_3}{2}}\Pi'^{J_1}_{n_1,m_1}\bar{\Pi}'^{J_2}_{n_2,m_2}O'^{J_3}_{n_3,m_3}. \end{aligned}$$

in agreement with cubic Supergravity interactions.

The sugra amplitudes are special in that they do not seem to receive corrections from the Yang-Mills interaction. This is clear for their energies (i.e. anomalous dimensions) where non-renormalization theorems have been obtained. Next we discuss the next set of operators of interest, which contain stringy excitations and consequently do receive corrections from the Yang-Mills interaction. Let us concentrate on the subset of the full loop space consisting of the gauge theory operators

$$\begin{aligned} \tilde{O}^J &= \sqrt{JN^J}O^J = \mathrm{Tr}(A^{\dagger J}), \\ \tilde{O}^J_n &= \sqrt{JN^J}O^J_n = \sum_{l=0}^{J} q^l \mathrm{Tr}(b^\dagger (A^\dagger)^l c^\dagger (A^\dagger)^{J-l}) = \sum_{l=0}^{J} q^l O_l. \end{aligned} \qquad (1)$$

Consider now the action of $\hat{H}$ on the two impurities state

$$
\begin{aligned}
\hat{H}\tilde{O}^J_n \quad &= -g^2_{YM}\Big(2N[\sum_{l=1}^{J} q^l(O^J_{l-1}-O^J_l)+\sum_{l=0}^{J-1} q^l(O^J_{l+1}-O^J_l)] \\
&+\sum_{l=2}^{J} q^l\sum_{l'=1}^{l-1}\tilde{O}^{l'}(O^{J-l'}_{l-l'-1}-O^{J-l'}_{l-l'})+\sum_{l=0}^{J-2} q^l\sum_{l'=0}^{J-l-2}\tilde{O}^{J-l-l'-1}(O^{l+l'+1}_{l+1}-O^{l+l'+1}_{l}) \\
&+\sum_{l=2}^{J} q^l\sum_{l'=0}^{l-2}\tilde{O}^{l-l'-1}(O^{J-l+l'+1}_{l'}-O^{J-l+l'+1}_{l'+1})+\sum_{l=0}^{J-2} q^l\sum_{l'=1}^{J-l-1}\tilde{O}^{l'}(O^{J-l'}_{l+1}-O^{J-l'}_{l}) \\
&+\sum_{l=0}^{J} q^l\tilde{O}^{J-l}(O^l_0-O^l_l)+\sum_{l=0}^{J} q^l\tilde{O}^{l}(O^{J-l}_{J-l}-O^{J-l}_0).
\end{aligned}
$$

By changing the order of the double summations, expressing $O_l=\sum_n e^{-\frac{2\pi inl}{J+1}}\tilde{O}^J_n/(J+1)$, performing the intermediate sums, taking into account the normalization of the states and taking the large J limit, we obtain ($y=\frac{J_1+1}{J+1}\to\frac{J_1}{J}$)

$$
\hat{H}|O^J_n\rangle=\lambda' 8\pi^2 n^2|O^J_n\rangle - g_2\lambda'\sum_{J_1+J_2=J}\sum_{m=-\frac{J_1}{2}}^{\frac{J_1}{2}}\frac{1}{\sqrt{J}}\sqrt{\frac{1-y}{y}}\Big(\frac{8m}{ny-m}\Big)\sin^2(\pi ny)|O^{J_1}_m O^{J_2}\rangle
$$

where λ' and g_2 have their usual meanings

$$
\lambda'=\frac{g^2_{YM}N}{J^2}, \qquad g_2=\frac{J^2}{N}.
$$

As expected, apart from the diagonal term which gives the first string tension correction to the anomalous dimension, the action of $\hat{H}$ results in a splitting of the loop into two loops.

Consider next the action of $\hat{H}$ on two loops $O^{J_1}_m O^{J_2}$. Apart from the diagonal term and the splitting into a 3 trace state, this will exhibit the effect of joining loops $O^{J_1}_m$ and O^{J_2} into a single BMN state $O^{J_1+J}_n$. This process is obtained when a derivative acts on $O^{J_1}_m$ and the other on O^{J_2}. Explicitly ($J=J_1+J_2$)

$$
\begin{aligned}
\hat{H}(\tilde{O}^{J_1}_m\tilde{O}^{J_2}) \quad &= -g^2_{YM}\Big[2N\Big(\sum_{l=0}^{J_1-1} q^l_m(O^{J_1}_{l+1}-O^{J_1}_l)+\sum_{l=1}^{J_1} q^l_m(O^{J_1}_{l-1}-O^{J_1}_l)\Big)\tilde{O}^{J_2} \\
&+2J_2\sum_{l=0}^{J_1} q^l_m[(O^J_{l+1}-O^J_l)+(O^J_{J_2+l-1}-O^J_{J_2+l})]+\texttt{3 trace states}\Big].
\end{aligned} \tag{2}
$$

Again by re-expressing O_l^J in terms of $\tilde{O}_n^J$, performing the intermediate sums, taking into account the normalization of the states and taking the large J limit we obtain

$$\hat{H}|O_m^{J_1}O^{J_2}\rangle = \lambda'\Big(\frac{8\pi^2 m^2}{y^2}\Big)|O_m^{J_1}O^{J_2}\rangle + \texttt{3 trace states} - \tfrac{\lambda' g_2}{\sqrt{J}}\sqrt{\tfrac{1-y}{y}}\Big(\tfrac{8ny}{m-ny}\Big)\sin^2(\pi ny)|O_n^J\rangle. \tag{3}$$

There is another set of 2 impurity traces

$$\tilde{O}_\phi^J = \sqrt{N^{J+1}}O_\phi^J = \mathrm{Tr}(b^\dagger A^{\dagger J}), \qquad \tilde{O}_\psi^J = \sqrt{N^{J+1}}O_\psi^J = \mathrm{Tr}(c^\dagger A^{\dagger J}).$$

for which our hamiltonian produces

$$\begin{aligned}\hat{H}(\tilde{O}_\phi^J\tilde{O}_\psi^{J_2}) &= -g_{YM}^2\Big(\sum_{l=0}^{J_2-1}\Big[(O_{J-l}^J - O_{J-l-1}^J) + (O_{J_2-l-1}^J - O_{J_2-l}^J)\Big] \\ &+ \sum_{l=0}^{J_2-1}\Big[(O_l^J - O_{l+1}^J) + (O_{l+J_2+l}^J - O_{J_2+l}^J)\Big]\Big).\end{aligned} \tag{4}$$

Altogether one has the following(collective) field theory Hamiltonian acting on the loop space

$$\begin{aligned}\hat{H} &= \lambda'(8\pi^2 n^2)O_n^J\frac{\partial}{\partial O_n^J} + \sum_{n,m,y}\lambda' g_2 D_{n,my}O_m^{J_1}O^{J_2}\frac{\partial}{\partial O_n^J} \\ &+ \sum_n \lambda' g_2 D_{my,n}O_n^J\frac{\partial}{\partial O_m^{J_1}}\frac{\partial}{\partial O^{J_2}} \\ &+ \lambda' g_2 D_{y,n}O_n^J\frac{\partial}{\partial O_\phi^{J_1}}\frac{\partial}{\partial O_\psi^{J_2}}.\end{aligned}$$

This contains anomalous dimensions and cubic interaction terms which are shown (see[3] to agree with the cubic couplings generated from the pp wave SFT .

3.1 Discretized String

The structure of more general states can be given in terms of a "lattice" string . We describe both of these in what follows:

Distinguished the Higgs components Z and $\bar{Z}$ from $\Phi_a, \bar{\Phi}_a$ $a = 1,2$ and also $D\phi$'s and DZ's. Let us concentrate on observables with insertions of

$$\Phi_a(l) \equiv Z^l \Phi_a Z^{-l}$$

Going to general "stringy" states

$$\mathrm{Tr}\left(\phi_{a_1}(l_1)\ \phi_{a_2}(l_2)\cdots\phi_{ak}(l_k)Z^J\right) \sim O$$

It is instructive to first exhibit the quadratic part of SFT:

$$H_2 = \Psi^+ \hat{h}\Psi(\{l\})$$

where the single string Hamiltonian

$$\hat{h}_{BMN} \sim \sum_l b^\dagger(l)b(l) + \frac{1}{\epsilon^2}\sum_l \left(b^\dagger_{l+1} + b_{l+1} - b^\dagger_l - b_l\right)^2$$

can be reproduced from Y-M theory:

$$H_{YM} = \mathrm{Tr}\left\{Z\frac{\partial}{\partial Z}\sum_a \Phi_a \frac{\partial}{\partial\phi_a}\,\phi_a\,\frac{\partial}{\partial\phi}_a\right\} + H_{int}$$

In the coherent state basis we have

$$Z \to A^\dagger\ ,\ \bar{Z} \to A$$

and likewise for the "impurity"

$$\Phi_a \to b^\dagger_a$$

$$H_1 \to g^2_{YM}\left(\left[b^\dagger, A^\dagger\right]\left[\frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger}\right] + \left[\frac{\partial}{\partial b^\dagger}, A^\dagger\right]\left[b^\dagger, \frac{\partial}{\partial A^\dagger}\right] + \left[b^\dagger, A^\dagger\right]\left[b^\dagger, \frac{\partial}{\partial A^\dagger}\right] + \left[\frac{\partial}{\partial b^\dagger}, \frac{\partial}{\partial A^\dagger}\right]\right),$$

The first two terms will conserve the number of imp. The last two will not.

consider the first (two) terms: Simple

$$\begin{aligned}
\mathrm{Tr}(bAA^\dagger b^\dagger) &\ :\ \mathrm{Tr}(\cdots b^\dagger A^\dagger\cdots) \Rightarrow N\mathrm{Tr}(\cdots b^\dagger A^\dagger\cdots) \\
\mathrm{Tr}(Abb^\dagger A^\dagger) &\ :\ \mathrm{Tr}(\cdots A^\dagger b^\dagger) \Rightarrow N\mathrm{Tr}(\cdots A^\dagger b^\dagger) \\
\mathrm{Tr}(bAb^\dagger A^\dagger) &\ :\ \mathrm{Tr}(\cdots A^\dagger b^\dagger\cdots) \Rightarrow N\mathrm{Tr}(\cdots b^\dagger A^\dagger\cdots)
\end{aligned}$$

We observe the action of $+0(1)$ splitting effects a permutation operator. We get

$$\hat{h}_{col} = \lambda\sum_{l=0}^{J=1}\left(\hat{b}^\dagger_{l+1}\hat{b}_{l+1} + \hat{b}^\dagger_l\hat{b}_l - 2\hat{b}^\dagger_{l+1}\hat{b}_l - 2b^\dagger_l b_l + \hat{b}^{\dagger 2}_{l+1} + b^{\dagger 2}_l - 2b^\dagger_l b^\dagger_{l+1} + b^2_{l+1} + b^2_l - 2\hat{b}_l\hat{b}_l\right)$$

comparison with BMN

$$\hat{h}_{coe} \Rightarrow \hat{h}_{BMN\ string} = \frac{1}{\epsilon^2} \sum_{l=0}^{J-1} \left(b_{l+1}^{\dagger} + b_{l+1} - b_l^{\dagger} - b_l \right)^2$$

for the case of Kunz type oscillators

$$b_l b_l^{\dagger} - b_l^{\dagger} b_l = 1$$

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4 SPIN CHAINS

The dynamics of lattice strings can be also equivalently represented in terms of a spin-chain. The trace

$$\mathrm{Tr}\left(b^\dagger(l_1)b^\dagger(l_2)\cdots b^\dagger(l_n)(A^\dagger)^J\right) \leftrightarrow b{l_1}^\dagger b^\dagger_{l_2}\cdots b^\dagger_{l_n}|O,J\rangle$$

becomes

$$\mathrm{Tr}\left(Z^{l_1}b^\dagger Z^{l_2-l_1}b^\dagger - b^\dagger Z^{J-l_n}\right) \to |\underbrace{ZZ\cdots Z}_{l_1}\, b^\dagger\, \underbrace{Z\cdots Z}_{l_2-l_1}\, b^\dagger\cdots b^\dagger Z\cdots Z\rangle$$

which in the spin-chain notation corresponds to spins $\pm\frac{1}{2}$ distributed on a lattice:

$$|++\cdots+-+++-\cdots\rangle$$

The (first quantized) string hamiltonian becomes a typical spin-chain hamiltonian with nearest neighbor coupling

$$\hat{h} = \lambda\sum_{i\neq j}(1-P)_{ij}) = \frac{\lambda}{4}\sum_{(ij)}\left(1-4\vec{S}_i\cdots\vec{S}_j\right)$$

It is useful to consider some specific examples: Take a two-body state defined by

$$\Psi = \sum_{1\leq l_1<l_2\leq L}\Psi(l_1,l_2)\mathrm{Tr}\left(\cdots Z\phi Z\cdots Z\phi Z\cdots\right)$$

Application of the Hamiltonian then results in the following eigen-equations:

$$\begin{aligned}
\text{for}\quad & l_2 > l_1+1: \\
& E\Psi(l_1,l_2) = 2\psi(l_1,l_2) - \Psi(l_1-1,l_2) - \Psi(l_1+1,l_2) + \\
& +2\Psi(l_1,l_2) - \Psi(l_1,l_2-1) - \Psi(l_1,l_2+1) \\
\text{for}\quad & l_2 = l_1+1: \\
& E\Psi(l_1,l_2) = 2\Psi(l_1,l_2) - \Psi(l_1-1,l_2) - \Psi(l_1,l_2+1).
\end{aligned}$$

This type of problem has a long history, it can be solved in terms of a Bethe ansatz

$$\Psi(l_1,l_2) = e^{ip_1l_1+ip_2l_2} + S_{p_2,p_1)}e^{ip_2l_1+ip_1l_2}\,.$$

where

$$p_k = \sum_{k=1}^{M} 4 \sin^2 \left(\frac{pk}{2}\right) ,$$

and the S-matrix is given by

$$S(p_1, p_2) = -\frac{e^{ip_1+ip_2} - 2e^{ip_1} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} + 1}$$

Consider next the insertion of fermionic operators

$$\mathrm{Tr}\, \psi^M Z^{J-M/2} + \cdots$$

where J denotes an R-charge and M the number of "impurities".

The resulting Hamiltonian action now leads to

$$h = \sum_{l=1}^{L} (1 - \Pi_{l,l+1})$$

and involves the graded permutation operator $\Pi_{l,l+1}$, it exchanges impurities at sites l and $l+1$, with a minus sign if the exchange involves two fermions. This Hamiltonian corresponds to a *free* lattice fermion. In terms of Pauli matrices $\sigma_l^1, \sigma_l^2, \sigma_l^3$:

$$h = \sum_{l=1}^{l} \left[1 - \sigma_l^3) - \frac{1}{2} \left(\sigma_l^1 \sigma_{l+1}^1 + \sigma_l^2 \sigma_{l+1}^2 \right) \right] .$$

The two-body states now obey the eigen-equations:

$$\begin{aligned} E\Psi(l_1, l_2) &= 2\Psi(l_1, l_2) - \Psi(l_1 - 1, l_2) - \Psi(l_1 + 1, l_2) + \\ &\quad +2\Psi(l_1, l_2) - \Psi(l_1, l_2 - 1) - \Psi(l_1, l_2 + 1) \,; l_2 > l_1 + 1 \end{aligned} \tag{5}$$

$$E\Psi(l_1, l_2) = 4\Psi(l_1, l_2) - \Psi(l_1 - l, l_2) - \Psi(l_1, l_2 + 1) \,;\, l_2 = l_1 + 1$$

These are identical to the ones of the $su(2)$ but for a different factor (2 instead of 4). The S-matrix is different. However,

$$S(p_1, p_2) = -1$$

reflects free fermi statistics. The momenta are

$$p_k = \frac{2\pi n_k}{L} ,$$

Next we consider states with derivative insertions. Denoting Z, DX, D^2Z as, $|0\rangle, |1|, |2\rangle$, one has the action of the Hamiltonian as

$$\begin{aligned}
&\mathcal{H}\cdot|1,0\rangle = |1,0\rangle - |0,1\rangle \quad \mathcal{H}\cdot|0,1\rangle = |0,1\rangle - 1,0\rangle \\
&\mathcal{H}\cdot|1,1\rangle = 2|1,1\rangle - |2,0\rangle - |0,2\rangle \\
&\mathcal{H}\cdot|2,0\rangle = \frac{3}{2}|2,0\rangle - |1,1\rangle - \frac{1}{2}|0.2\rangle \quad \mathcal{H}\cdot|0,2\rangle = \frac{3}{2}|0,2\rangle - |1,1\rangle - \frac{1}{2}|2,0\rangle \,,
\end{aligned}$$

On two-body traces of the form

$$\Psi = \sum_{1\le l_1 \le l_2 \le L} \Psi(l_1,l_2)\,\mathrm{Tr}\,(\cdots Z(DZ)Z\cdots Z(DZ)Z\cdots)\,,$$

the Hamiltonian becomes

$$\begin{aligned}
E\Psi(l_1,l_2) \qquad &= 2\Psi(l_1,l_2) - \Psi(l_1-l,l_2) - \Psi(l_1+l,l_2) + \\
&+2\Psi(l_1,l_2) - \Psi(l_1,l_2-1) - \Psi(l_1,l_2+1)\,;\, l_2 > l_1
\end{aligned}$$

$$\begin{aligned}
E\Psi(l_1,l_2 \qquad &= \frac{3}{2}\Psi(l_1,l_2) - \Psi(l_1-1,l_2) - \frac{1}{2}\Psi(l_1-l,l_2-1) + \\
&+\frac{3}{2}\Psi(l_1,l_2) - \Psi(l_1,l_2+1) - \frac{1}{2}\Psi(l_1+l,l_2+1)\,;\, l_2 = l_{1+1} \quad (6)
\end{aligned}$$

with the following solution for the S-matrix

$$S''(p_1,p_2) = -e^{ip_1+ip_2} - 2e^{ip_2} + 1$$

4.0.1 Comparison of the Lattice and Spin-Chain Models

It is instructive to do a comparison between the spin-chain analysis and the lattice string (of length J) and the Hamiltonian

$$H = -\sum_{i=1}^{J}\left[b^\dagger_{i+1}b_i + b^\dagger_i b)i + 1 - b^\dagger_i b_i - b^\dagger_{i+1}b_{i+1}\right]\,.$$

Examples

- $n = 1$

In this trivial case we should consider $N = J+1, n=1$. Let us first look at the general eigenstates without the constraints. In the lattice case we have that this state is given by

$$\begin{aligned} |\psi_l\rangle &= \sum_{i=0}^{J} e^{\frac{2\pi l}{J}} b_i^\dagger |0\rangle \\ H|\psi_l &= 4\sin^2\left(\frac{\pi}{J} l\right) |\psi_l\rangle, l = 0, \cdots, J-1 \end{aligned}$$

In the spin chain case with $N = J+1$ we have that an eigenstate with $Q|\psi_n\rangle = J|\psi_n\rangle$ is given by a similar expression

$$\begin{aligned} |\psi_l\rangle &= \sum_{i=0}^{J+2} e^{\frac{2\pi l}{J+1} i} S_i^\dagger |0\rangle \\ H|\psi_l\rangle &= \sin^2\left(\frac{\pi}{J+1} l\right) |\psi_l\rangle, l = 0, \cdots, J \end{aligned}$$

In this example we see that without the constraint $l = n = 0$, we can not claim agreement (the reason is the different periodicity of the two problems). Locally the two problems look the same, the two Hamiltonians act in the same way.

- n=2

This time we should take $N = J + 2Q = J$. Let us first look at the lattice case, we consider the Z_J invariant state

$$|\psi_n\rangle = \sum_l \sum_{k=0}^{J} q_n^k b_l^\dagger b_{l+k}^\dagger .$$

We leave $q_n =$ unspecified for now. We will see that the presence of not regular oscillators will force us to pick $q_n = e^{\frac{i\pi n}{K+1}}$ instead of $q_n = e^{\frac{i2\pi n}{J}}$ which is something that is probably expected for a J periodic problem. The action of the Hamiltonian is given by

$$\begin{aligned} & H|\psi_n\rangle + \\ & -\sum_l b_l^\dagger \Bigg[2q_m^{-1} \sum_{k=2}^{J} b_{l+k}^\dagger + 2q_n \sum_{k=0}^{J=2} q_n^k b_{l+k}^\dagger \\ & -4 \sum_{k=l}^{J-1} q_n^k b_{l+k}^\dagger + 2(1+q_n^J)\left(b_l^\dagger b) l + 1^\dagger - b_l^\dagger b_l^\dagger\right) \Bigg] |0\rangle \end{aligned}$$

From the above we see that in order for $|\psi_n\rangle$ to be an eigenstate, one should set $q_n = e^{\frac{i2\pi n}{J+1}}$. After this comment we see that

$$H|\psi_n\rangle = 4\sin^2\left(\frac{\pi n}{J+1}\right)|\psi_n\rangle \, .$$

This is the same as for the spin chain state of the form

$$|\phi_n\rangle = \sum_l \sum_{k=1}^{J+1} S_l^\dagger s_{l+k}^\dagger q_n^k |0\rangle \, ,$$

with the same values for q_n . After the quantization of q's we see agreement between the two pictures.

In the above examples we have worked only to leading order in the Yang -Mills coupling constant. A large number of studies were(and are being) performed addressing the question of higher loop corrections[1]. There are encouraging indication that an all order solution might be forthcoming.

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99